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A relation between $N = 8$ gauge theories in three dimensions

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Abstract: We show that three-dimensional $N = 8$ $Sp(N)$ and $SO(2N+1)$ gauge theories flow to the same strong coupling fixed point. As a consequence, the corresponding orientifold two-planes in type IIA string theory are described at strong coupling and low-energies by the same M theory background. In the large N limit, these assertions are confirmed by studying discrete torsion in the supergravity theory corresponding to membranes on $\mathbb{R}^8/\mathbb{Z}_2$.

Keywords: Field Theories in Lower Dimensions, Duality in Gauge Field Theories, M-Theory, Supersymmetry and Duality.

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Contents

1. $N = 8$ gauge theories in three dimensions

In four dimensions, $N = 4$ Yang-Mills theories provide some of the simplest examples of interacting conformal field theories. The existence of S duality implies that there are at least two distinct realizations of the same conformal field theory [1]. The purpose of this letter is to explain a similar identification between conformal field theories in three dimensions. Our starting point is $N = 8$ Yang-Mills theory in three dimensions with gauge group G.[1] The coupling constant, g^2, has dimension one in three dimensions. For simplicity, we will assume there is only one coupling constant. To obtain an interacting conformal field theory, we must therefore consider the infra-red limit of the theory where $g^2 \to \infty$ so the theory is strongly coupled.

Let the gauge group G have rank N. The gauge theory contains seven scalar fields, ϕ^i, in the adjoint representation of G. The moduli space of the theory is

$$\mathcal{M} = \frac{\mathbb{R}^{7N} \times \widehat{T}^N}{\mathcal{W}}, \tag{1.1}$$

where $\mathcal{W}$ is the Weyl group of G and $\widehat{T}^N$ is the Cartan torus for the dual group $\widehat{G}$ [2]. The manifest R-symmetry of the theory is $Spin(7)$. The compact directions in the moduli space $\mathcal{M}$ correspond to the expectation values of the scalar fields, σ, dual to the N massless photons present at generic points in $\mathcal{M}$. The size of the compact directions in $\mathcal{M}$ is proportional to g^2 and in the strong coupling limit, $\widehat{T}^N \to \mathbb{R}^N$.

A non-trivial conformal field theory can only appear at a singularity of the moduli space $\mathcal{M}$ in the strong coupling limit. For the case where $G = U(N)$, arguments given in [3, 4, 2] showed that the theory at the origin of the moduli space flows to an interacting $Spin(8)$ invariant fixed point. Note that there are other singularities in the moduli space but we will focus on the theory at the origin.

There are two ways to show $Spin(8)$ invariance. One argument only involves supersymmetry and the conformal group in three dimensions, which is isomorphic to

[1] Both four-dimensional $N = 4$ Yang-Mills and three-dimensional $N = 8$ Yang-Mills have sixteen real supersymmetries.

$Spin(3,2)$. Closure of the superconformal algebra with sixteen supersymmetry generators simply requires a $Spin(8)$ symmetry [5, 2]. This argument is true for any gauge group G.

The second argument for $Spin(8)$ invariance will provide additional information. This argument uses the strong-weak coupling duality of four-dimensional gauge theory. Let us start with $N = 4$ Yang-Mills in four dimensions with gauge group G and coupling constant τ. This theory has a dual realization in terms of a theory with gauge group $\widehat{G}$ and coupling $-1/\tau$. We can set the theta angle to zero for our purposes. We can then restrict to the $\mathbb{Z}_2$ subgroup of the $SL(2,\mathbb{Z})$ strong-weak coupling duality which relates the dual coupling $\tilde{\lambda}$ to the original four-dimensional coupling constant λ in the following way:

$$\tilde{\lambda} = \frac{2\pi}{\lambda}\,. \tag{1.2}$$

Let us compactify the gauge theory on $\mathbb{R}^3 \times S^1$. The effective three-dimensional coupling constant is given by

$$g^2 = \frac{\lambda^2}{2\pi R}\,, \tag{1.3}$$

where R is the radius of the circle. The component A of the four-dimensional gauge field along S^1 gives an extra adjoint-valued scalar

$$\phi_e = \frac{1}{2\pi R}\int_{S^1} A\,. \tag{1.4}$$

This scalar field has a compact moduli space with a size proportional to $1/R$. In the $R \to 0$ limit with g^2 held fixed, this scalar field becomes $Spin(7)$ symmetric with the six scalars of the four-dimensional theory.

In addition, we have another N scalars, ϕ_m, coming from dualizing the three-dimensional photons. The moduli space for these scalars, which correspond to the choice of 't Hooft lines or magnetic Wilson lines along S^1, is also compact with a size again proportional to $1/R$. In the $R \to 0$ limit with λ held fixed, these scalars become rotationally symmetric with the remaining non-compact scalars. Electric-magnetic duality then exchanges G with $\widehat{G}$ and exchanges the ϕ_e and ϕ_m directions.

When G and $\widehat{G}$ are different,[2] this duality action is not a symmetry of the theory for any finite value of the three-dimensional coupling constant. Rather, the $\mathbb{Z}_2$ duality identifies one theory with a different theory. In the infra-red limit where $g^2 \to \infty$, we can combine this duality with the known $Spin(8)$ invariance. The $Spin(8)$ invariance includes rotation of ϕ_e into ϕ_m but does not exchange G with $\widehat{G}$. We can therefore conclude that gauge theories in three dimensions with either gauge group G or gauge group $\widehat{G}$ flow to the same $Spin(8)$ invariant superconformal field theory.

[2] We are only interested in differences at the level of the Lie algebra rather than the global structure of the group. For example when $G = U(N)$, this $\mathbb{Z}_2$ action is a symmetry if we choose the self-dual value for the coupling [3].

2. Orientifolds and supergravity

The argument given in section 1 is primarily interesting for the non-simply-laced gauge groups $Sp(N)$ and $SO(2N+1)$. For any finite coupling constant, these two theories are quite different but they flow in the infra-red to the same conformal field theory at the origin of the moduli space. These gauge theories appear naturally in type IIA string theory as the low-energy excitations of N D2-branes coincident with various orientifold two-planes. Our starting point is then type IIA string theory on the orientifold $\mathbb{R}^7/\mathbb{Z}_2$. The gauge theory coupling for D2-branes on top of the orientifold plane is related to the string coupling g_s and string scale M_s by

$$g^2 = g_s M_s \,. \tag{2.1}$$

The $\mathbb{Z}_2$ action must act on the circle on which we reduce from M theory to type IIA if the moduli space seen by a D2-brane probing the orientifold is to agree with (1.1) for the gauge theory [6, 7]. When compactified on

$$\frac{\mathbb{R}^7 \times S^1}{\mathbb{Z}_2}\,, \tag{2.2}$$

the Chern-Simons interaction in eleven-dimensional supergravity

$$-\frac{1}{6}\int C \wedge G \wedge G \tag{2.3}$$

is only invariant if C is invariant under the $\mathbb{Z}_2$ action. The M theory lift of the orientifold is therefore the $\mathbb{Z}_2$ orbifold (2.2). The gauge theory coupling can be expressed in terms of the eleven-dimensional scale, M_{pl}, and the size R_{11} of the S^1 in (2.2):

$$g^2 = M_{pl}^3 R_{11}^2 \,.$$

As usual, to restrict to the field theory modes on the branes, we take the limit $M_{pl} \to \infty$ holding fixed g^2. The strong coupling limit for the gauge theory then corresponds to decompactifying the circle on which we reduce from M theory to type IIA.

We will distinguish between three kinds of orientifold two-planes. The gauge symmetry of N D2-branes on top of the orientifold plane $O2^-$ is $SO(2N)$. We will count branes and charges on the quotient space. This O2-plane carries $-1/8$ units of membrane charge. We can also have a stuck $1/2$ membrane at the orientifold fixed point. We will call this case $\widetilde{O2}^+$. In this case, the gauge symmetry is $SO(2N+1)$ and the charge is $3/8$. Lastly, we will also consider $O2^+$ which corresponds to the choice of Chan-Paton factors giving gauge group $Sp(N)$. This O2-plane has charge $1/8$.[3]

[3] We will not distinguish cases where the Ramond gauge-field has a non-trivial holonomy around the circle in (2.2). This holonomy was important in the case of $O4$-planes studied in [8, 9]. In those cases, the limit where $R_{11} \to \infty$, which we will primarily study, was not necessarily smooth. For example, $O4^0$ only exists with a finite size circle. In this respect, the case of $O2$-planes is nicer since the gauge theory has a smooth strong coupling limit so we can take the limit where $R_{11} \to \infty$. In this limit, any holonomy involving the circle goes away.

Let us start with the M theory lift of $O2^-$. There are two singularities in (2.2) at 0 and π on the circle. At strong coupling, the geometry around either singularity becomes $\mathbb{R}^8/\mathbb{Z}_2$ and the D2-branes become membranes which we can place at the fixed point. To compute the membrane charge of this configuration, we can consider M theory on $T^8/\mathbb{Z}_2$. There is a net membrane charge in this theory coming from the interaction [10, 11]

$$-\int C \wedge X_8(R) \tag{2.4}$$

in M theory. The total membrane charge is given by -$\chi/24$ and must be cancelled to avoid a tadpole anomaly [12]. Although the space $T^8/\mathbb{Z}_2$ is singular, standard string theory technology for orbifolds can be used to show that $\chi/24 = 16$. Each fixed point, and consequently the $\mathbb{R}^8/\mathbb{Z}_2$ fixed point, is then a source of $-1/16$ units of membrane charge [13, 6].

To count the distinct M theory configurations corresponding to the remaining O2-planes, we turn to the supergravity solutions dual to these conformal field theories [14]. The supergravity solutions for the various O3-planes were analyzed in [15], while membranes on the orbifold space $\mathbb{R}^8/\mathbb{Z}_2$ were mentioned in [16]. The $SO(2N)$ $(2,0)$ theory was considered from a supergravity perspective in [17]. M theory compactified on $AdS_4 \times \mathbb{R}P^7$ is dual to the world-volume theory of membranes on $\mathbb{R}^8/\mathbb{Z}_2$ in the limit where $M_{pl} \to \infty$. We wish to count the number of distinct fluxes for the four-form G on $\mathbb{R}P^7$. This counts the number of distinct strong coupling limits for $O2$-planes. Note that since $w_4 = 0$ for $\mathbb{R}^8/\mathbb{Z}_2$, no half-integral G flux is possible [18].

The counting of fluxes is then rather simple. We need to compute $H^4(\mathbb{R}P^7, \mathbb{Z})$ which, using the results of [15], is $\mathbb{Z}_2$. Therefore, there is only one possible choice of discrete torsion. This is a supergravity confirmation of the argument in section 1. The low-energy theories on the three orientifold two-planes, $O2^-, \widetilde{O2}^+$ and $O2^+$ must flow to two distinct strong coupling conformal field theories. We know that $O2^-$ flows to the case without torsion so $\widetilde{O2}^+$ and $O2^+$ must flow to the case with torsion.

In the case where we turn on discrete torsion, we can ask: by how much does the membrane charge change? The charge shift is generated by the term (2.3) and is determined by computing

$$-\frac{1}{2}\int_{\mathbb{R}P^7} \frac{G}{2\pi} \wedge \frac{C}{2\pi}\,, \tag{2.5}$$

where G is the torsion class. To compute this integral, we will view $\mathbb{R}P^7$ as the boundary of a smooth eight-dimensional space $\mathcal{M}$.[4] We can then evaluate

$$-\frac{1}{2}\int_{\mathcal{M}} \frac{G}{2\pi} \wedge \frac{G}{2\pi}\,,$$

rather than (2.5). Let us begin by recalling the construction of the Hopf fibration of S^7 over $\mathbb{C}P^3$. The sphere S^7 is the locus of points in $\mathbb{C}^4$ obeying

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1\,,$$

[4] I am especially grateful to E. Witten for a detailed explanation about how to compute this integral.

where $\vec{z} = (z_1, z_2, z_3, z_4)$ coordinatize $\mathbb{C}^4$. To obtain $\mathbb{C}P^3$ from S^7, we quotient by the $U(1)$ action

$$\vec{z} \sim e^{i\alpha} \vec{z}\,. \tag{2.6}$$

The fibers over $\mathbb{C}P^3$ are then circles. We can describe this construction in a slightly different way. Let us start with the total space of the bundle $O(-1)$ over $\mathbb{C}P^3$. Let w be a coordinate for the fiber, which is a copy of $\mathbb{C}$. We obtain a smooth eight manifold by taking a disk $|w| < 1$ in each fiber over $\mathbb{C}P^3$. The boundary of this space is S^7. We can mimick this construction for the case of $\mathbb{R}P^7$ by taking the bundle $O(-2)$ rather than $O(-1)$. This gives us the eight manifold $\mathcal{M}$.

Since $\mathbb{R}P^7$ is orientable, we can associate a homology class in $H_3(\mathbb{R}P^7, \mathbb{Z})$ to G using Poincaré duality. This class can be represented by an $\mathbb{R}P^3$ subspace of $\mathbb{R}P^7$ which is the boundary of a four-cycle $\mathcal{W}$ in $\mathcal{M}$. Let $\pi : \mathcal{M} \to \mathbb{C}P^3$ denote the projection map. The Hopf bundle $\mathbb{R}P^3 \to \mathbb{C}P^1$ is compatible with the projection map π in the sense that $\pi(\mathcal{W})$ is a $\mathbb{C}P^1$ in $\mathbb{C}P^3$.

The connection C for the torsion class G obeys

$$\int_{\partial\mathcal{W}} \frac{C}{2\pi} = \frac{1}{2} \tag{2.7}$$

and therefore

$$\int_{\mathcal{W}} \frac{G}{2\pi} = \frac{1}{2}\,. \tag{2.8}$$

We need to identify a class G in $\mathcal{M}$ obeying (2.8). The Poincaré dual of the base $\mathbb{C}P^3$ of the bundle $\mathcal{M}$ is a two-form X satisfying

$$\int_{\mathcal{W}} X \wedge X = -2\,.$$

We can then take $G/2\pi = -X^2/4$. The charge shift is then given by

$$-\frac{1}{2(2\pi)^2} \int_{\mathcal{M}} G \wedge G = -\frac{1}{16} \int_{\mathcal{M}} X^4 = \frac{1}{4}\,. \tag{2.9}$$

The final integral is evaluated by integrating over one X which restricts the remaining integral to the base $\mathbb{C}P^3$, which is then standard.

Turning on the discrete torsion therefore shifts the membrane charge from $-1/16$ to $3/16$. There is an amusing interpretation for this $1/4$ membrane charge. $\widetilde{O2}^+$ differs from $O2^-$ by a stuck $1/2$ membrane. In studying the strong coupling description of $\widetilde{O2}^+$, we see that the orientifold plane splits into two singularities from (2.2). Apparently the $1/2$ stuck membrane also splits into two fluxes, each carrying $1/4$ unit of membrane charge. The case of $O2^+$ is a little different. The singularity at the origin of (2.2) has charge $3/16$ but the other singularity at π should have charge $-1/16$ since the total membrane charge for $O2^+$ is $1/8$.

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